MAT185 Linear Algebra
Assignment 4

Instructions:

Please read the **MAT185 Assignment Policies & FAQ** document for details on submission policies, collaboration rules and academic integrity, and general instructions.

1. **Submissions are only accepted by** Gradescope. Do not send anything by email. Late submissions are not accepted under any circumstance. Remember you can resubmit anytime before the deadline.
2. **Submit solutions using only this template pdf.** Your submission should be a single pdf with your full written solutions for each question. If your solution is not written using this template pdf (scanned print or digital) then your submission will not be assessed. Organize your work neatly in the space provided. Do not submit rough work.
3. **Show your work and justify your steps** on every question but do not include extraneous information. Put your final answer in the box provided, if necessary. We recommend you write draft solutions on separate pages and afterwards write your polished solutions here on this template.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

Academic Integrity Statement:

Full Name: ANUSHA FATIMA ALAM

Student number: 1009056539

Full Name: ______

Student number: ______

I confirm that:

- I have read and followed the policies described in the document **MAT185 Assignment Policies & FAQ.**
- In particular, I have read and understand the rules for collaboration, and permitted resources on assignments as described in subsection II of the the aforementioned document. I have not violated these rules while completing and writing this assignment.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while completing and writing this assignment.

By signing this document, I agree that the statements above are true.

Signatures: 1) ______

2) ______

1. Consider the sequence $\left\{\frac{1}{1}, \frac{3}{2}, \frac{7}{3}, \ldots, \frac{a_n}{b_n}, \ldots\right\}$ where $a_{n+1} = a_n + 2b_n$ and $b_{n+1} = a_n + b_n$.

(a) Find a matrix A such that $A\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix}$.

In order for the equation to be dimensionally consistent, we need to find a 2 by 2 matrix A, such that

$$\underset{2\times 2}{A}\underset{2\times 1}{\begin{bmatrix} a_n \\ b_n \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} a_n + 2b_n \\ a_n + b_n \end{bmatrix}}$$

$\boxed{\text{The matrix A that satisfies this condition is } \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.}$

(b) Find an invertible matrix S and a diagonal matrix D such that $A = SDS^{-1}$.

To find an invertible matrix S and a diagonal matric D such that $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = SDS^{-1}$, we can use the principle of diagonalization.

First, we need to determine the eigenvalues of matrix A:

The characteristic polynomial of matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ is given by:

$$\begin{vmatrix} 1-\lambda & 2 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) - 2 = \lambda^2 - 2\lambda - 1$$

We can determine the roots of this quadratic polynomial using the quadratic formula:

$$\lambda = \frac{2 \pm \sqrt{2^2+4}}{2} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

Therefore, the eigen-values of $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ are $1+\sqrt{2}$ and $1-\sqrt{2}$

We now, need to find the corresponding eigenvector for each eigenvalue:

For $\lambda = 1+\sqrt{2}$, $\left(\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} (1+\sqrt{2}) & 0 \\ 0 & (1+\sqrt{2}) \end{bmatrix}\right) X = 0$

$\downarrow$

So the eigenvalue of $\lambda = 1+\sqrt{2}$ has an eigenvector $\begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$.

For $\lambda = 1-\sqrt{2}$, $\left(\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} (1-\sqrt{2}) & 0 \\ 0 & (1-\sqrt{2}) \end{bmatrix}\right) X = 0$

$\downarrow$

So the eigen-vector for $\lambda = 1-\sqrt{2}$ is $\begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$.

We can now form matrix by placing eigenvectors as its columns: $S = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix}$

To form the diagonal matrix D, we can place the eigenvalues along the diagonal.

Matrix $D = \begin{bmatrix} (1+\sqrt{2}) & 0 \\ 0 & (1-\sqrt{2}) \end{bmatrix}$

$\boxed{\therefore \text{ Therefore } S = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} (1+\sqrt{2}) & 0 \\ 0 & (1-\sqrt{2}) \end{bmatrix}}$

We can verify this by computing SDS^{-1} for which we get $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ which is equivalent to matrix A.

1. Consider the sequence $\left\{\frac{1}{1}, \frac{3}{2}, \frac{7}{3}, \ldots, \frac{a_n}{b_n}, \ldots\right\}$ where $a_{n+1} = a_n + 2b_n$ and $b_{n+1} = a_n + b_n$.

(c) Use your answer from part (b) to find explicit formulas for a_n and b_n, and then show that $\lim_{n\to\infty} \frac{a_n}{b_n} = \sqrt{2}$.

From part (b) we know that $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (1+\sqrt{2}) & 0 \\ 0 & (1-\sqrt{2}) \end{bmatrix} \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix}^{-1}$.

We also know that $\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = A \begin{bmatrix} a_n \\ b_n \end{bmatrix}$. Therefore $\begin{bmatrix} a_n \\ b_n \end{bmatrix} = A \begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix}$.

Considering the n^{th} terms of the sequence and the relationship $\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = A \begin{bmatrix} a_n \\ b_n \end{bmatrix}$:

$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = A^0 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix} = A \begin{bmatrix} 3 \\ 2 \end{bmatrix} = A\left(A \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = A^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vdots$$

Therefore, $\begin{bmatrix} a_n \\ b_n \end{bmatrix} = A^n \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$= (SDS^{-1})^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} = SD^nS^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ because $A = SDS^{-1}$ [Part (b)]

$$= \left(\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (1+\sqrt{2})^n & 0 \\ 0 & (1-\sqrt{2})^n \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2\sqrt{2}} & \frac{1}{2} \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Hence $$\boxed{\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} \dfrac{(1+\sqrt{2})^n - (1-\sqrt{2})^n + \sqrt{2}((1+\sqrt{2})^n - (1-\sqrt{2})^n)}{2} \\ \dfrac{(1+\sqrt{2})^n - (1-\sqrt{2})^n + \sqrt{2}((1+\sqrt{2})^n + (1-\sqrt{2}^n))}{2\sqrt{2}} \end{bmatrix}.}$$

To determine $\lim_{n\to\infty} \frac{a_n}{b_n}$, we need to find a simplified form for $\frac{a_n}{b_n}$.

$$\frac{a_n}{b_n} = \frac{\sqrt{2}(1+\sqrt{2})^n + \sqrt{2}(1-\sqrt{2})^n + 2(1+\sqrt{2})^n - 2(1-\sqrt{2})^n}{(1+\sqrt{2})^n - (1-\sqrt{2})^n + \sqrt{2}(1+\sqrt{2})^n + \sqrt{2}(1-\sqrt{2})^n}$$

$$= \frac{(\sqrt{2}+2)(1+\sqrt{2})^n + (\sqrt{2}-2)(1-\sqrt{2})^n}{(\sqrt{2}+1)(1+\sqrt{2})^n + (\sqrt{2}-1)(1-\sqrt{2})^n} = \frac{(\sqrt{2}+2)(1+\sqrt{2})^n + (\sqrt{2}-2)(1-\sqrt{2})^n}{(\sqrt{2}+1)^{n+1} + (\sqrt{2}-1)(1-\sqrt{2})^n}$$

$$= ((\sqrt{2}+2)(1+\sqrt{2})^n + (\sqrt{2}-2)(1-\sqrt{2})^n)(\sqrt{2}+1)^{-n-1}$$

$$= ((\sqrt{2}+2)(1+\sqrt{2})^n)(\sqrt{2}+1)^{-n-1} = \frac{\sqrt{2}+2}{\sqrt{2}+1} = \frac{\sqrt{2}(1+\sqrt{2})}{(\sqrt{2}+1)} = \sqrt{2}$$

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \sqrt{2} = \sqrt{2}$, we know $\lim_{n\to\infty} \frac{a_n}{b_n}$ converges to $\sqrt{2}$

2. Let A be an $n \times n$ matrix, and suppose that the only eigenvalues of A are 0, 1, and 2.

(a) Prove that $\dim E_1(A) + \dim E_2(A) \leq \operatorname{rank} A$.

According to the rank-nullity theorem, we know that for a linear transformation $T: V \to W$, the dimension of the kernel (nullity) plus the dimension of the image space (rank) equals the dimension of the domain:

$$\dim \operatorname{Ker}(T) + \dim \operatorname{Im}(T) = \dim V$$

where V and W are finite dimensional.

For a matrix, the linear transformation is given by multiplication with A. i.e. $A: \mathbb{R}^n \to \mathbb{R}^n$. We can therefore apply the rank-nullity theorem to matrix A:

$$\dim \operatorname{Ker}(A) + \dim \operatorname{Im}(A) = n$$

where n is ~~the~~ is the number of columns of A.

Considering the eigenspaces of A, we know:

$$E_0(A) = \operatorname{Ker}(A)$$
$$E_1(A) = \{v \in \mathbb{R}^n \mid Av = v\}$$
$$E_2(A) = \{v \in \mathbb{R}^n \mid Av = 2v\}$$

Since the only possible eigenvalues for A are 0, 1, and 2 (as given), we know that the eigenspaces $E_0(A)$, $E_1(A)$, $E_2(A)$ are linearly independent if their dimensions add upto n:

$$\dim E_0(A) + \dim E_1(A) + \dim E_2(A) = n$$

If they aren't linearly independent then $\dim E_0(A) + \dim E_1(A) + \dim E_2(A) < n$

We want to show that $\dim E_1(A) + \dim E_2(A) \leq \operatorname{rank}(A)$.

In order to do so, we can use the rank-nullity theorem to rewrite this as:

$$\dim \operatorname{Ker}(A) + \dim \operatorname{Im}(A) = \dim \operatorname{Ker}(A) + \operatorname{rank}(A) = n$$
$$\geq \dim E_0(A) + \dim E_1(A) + \dim E_2(A).$$

$$\therefore \quad \dim \operatorname{Ker}(A) + \operatorname{rank}(A) \geq \dim E_0(A) + \dim E_1(A) + \dim E_2(A)$$

Since $\dim E_0(A) = \dim \operatorname{Ker}(A) = n - \operatorname{rank}(A)$ we know:

$$\operatorname{rank}(A) \geq \dim E_1(A) + \dim E_2(A)$$

Rearranging this we get

$$\dim E_1(A) + \dim E_2(A) \leq \operatorname{rank}(A)$$

2. Let A be an $n \times n$ matrix, and suppose that the only eigenvalues of A are 0, 1, and 2.

(b) Prove that if $\dim E_1(A) + \dim E_2(A) = \operatorname{rank} A$, then A is diagonalizable.

From part (a) we know that $\dim \operatorname{Im}(A) + \dim \operatorname{Ker}(A) = n$

$\therefore \quad \operatorname{rank}(A) + \dim \operatorname{Ker}(A) = n$

From part (a) we also know that if the eigenspaces $E_0(A)$, $E_1(A)$ and $E_2(A)$ are linearly independent and their dimensions add up to n; then:

$$\dim E_0(A) + \dim E_1(A) + \dim E_2(A) = n$$
$$= \dim \operatorname{Ker} A + \operatorname{rank}(A)$$

This means that $\dim E_1(A) + \dim E_2 = \operatorname{rank}(A)$ only if its eigenspaces are linearly independent. Therefore we would like to prove that the eigen vectors are linearly independent:

Proof:

Let $x_1, x_2, \dots, x_K$ be eigenvectors corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \dots \lambda_K$ of A.

Assume that $\{x_1, x_2, \dots x_K\}$ are linearly dependent. We can than find j such that $\{x_1, x_2, \dots, x_{j-1}\}$ is linearly independent, and $\{x_1, x_2, \dots, x_j\}$ is linearly dependent. Then we have:

$$[*] \quad a_1 x_1 + a_2 x_2 + \dots + a_j x_j = 0$$

where not all a_i's are zero, and in particular $a_j \neq 0$.

Multiplying $[*]$ by A from the left we get:

$$a_1 \lambda_1 x_1 + a_2 \lambda_2 x_2 + \dots + a_j \lambda_j x_j = 0$$

On the other hand, multiplying $[*]$ by λ_j, we can obtain:

$$a_1 \lambda_j x_1 + a_2 \lambda_j x_2 + \dots + a_j \lambda_j x_j = 0.$$

Subtracting the two equations we get

$$a_1(\lambda_1 - \lambda_j)x_1 + a_2(\lambda_2 - \lambda_j)x_2 + \dots + a_{j-1}(\lambda_{j-1} - \lambda_j)x_{j-1} = 0$$

and $a_1(\lambda_1 - \lambda_j) = a_2(\lambda_2 - \lambda_j) = \dots = a_{j-1}(\lambda_{j-1} - \lambda_j) = 0$

Since λ_i's are distinct, we have:

$$a_1 = a_2 = \dots = a_{j-1} = 0$$

and $a_j x_j = 0$, $a_j = 0$, a contradiction.

Therefore $\{x_1, x_2, \dots x_K\}$ is linearly independent.

Since we know that the eigenvectors are linearly independent when $\dim E_1(A) + \dim E_2(A) = \operatorname{rank}(A)$ then A is diagonalizable by the diagonalization theorem.